

Solution of a Yang–Baxter system

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Abstract

Yang–Baxter system related to quantum doubles is introduced and large class of both continuous and discrete symmetries of the solution manifold are determined. Strategy for solution of the system based on the symmetries is suggested and accomplished in the dimension two. The complete list of invertible solutions of the system is presented.

1 Introduction

The Yang–Baxter equations, both constant and spectral dependent proved to be an important tool for various branches of theoretical physics. They represent a system of N^6 cubic equations for elements of $N^2 \times N^2$ matrix R and can be written in the well known form

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \quad (1)$$

Even though many solutions are known for all types of the Yang–Baxter equations [1, 2, 3, 4, 5], until now the complete solution is known only for the constant Yang–Baxter equations in the dimension two [6], i.e. matrices 4×4 .

Various extensions of the Yang–Baxter equations for several matrices, called Yang–Baxter systems, appeared in literature [7, 8, 9]. The constant systems are used mainly for construction of special Hopf algebras while the spectral dependent solutions are applied in quantum integrable models. Examples of both types together with their particular solutions were presented in [10].

As the Yang–Baxter systems usually contain several Yang–Baxter–type equations, it is convenient to introduce the following notation: *The Yang–Baxter*

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commutator $[R, S, T]$ of (constant) $N^2 \times N^2$ matrices R, S, T is $N^3 \times N^3$ matrix

$$[R, S, T] := R_{12}S_{13}T_{23} - T_{23}S_{13}R_{12}. \quad (2)$$

In this notation, the constant Yang–Baxter equation is written as

$$[R, R, R] = 0, \quad (3)$$

while e.g. the system for quantized braided groups [9] reads

$$[Q, Q, Q] = 0, \quad [R, R, R] = 0, \quad (4)$$

$$[Q, R, R] = 0, \quad [R, R, Q] = 0. \quad (5)$$

The complete set of invertible solutions of this system in the dimension two is given in [11].

The goal of the present paper is to give a complete solution of another, more complicated constant system related to quantum doubles i.e. special quasitriangular Hopf algebras constructed from the tensor product of Hopf algebras by defining a pairing between them.

2 WXZ system and its symmetries

In the paper [12] a method of obtaining the quantum doubles for pairs of FRT quantum groups is presented: Let two quantum groups are given by relations [13]

$$W_{12}U_1U_2 = U_2U_1W_{12}$$

$$Z_{12}T_1T_2 = T_2T_1Z_{12}$$

where W and Z are matrices $N^2 \times N^2$ satisfying the Yang–Baxter equations

$$[W, W, W] = 0, \quad (6)$$

$$[Z, Z, Z] = 0, \quad (7)$$

and suppose that there is a matrix X that satisfies the equations

$$[W, X, X] = 0, \quad (8)$$

$$[X, X, Z] = 0, \quad (9)$$

Then the relations

$$X_{12}U_1T_2 = T_2U_1X_{12} \quad (10)$$

define quantum double with the pairing

$$\langle U_1, T_2 \rangle = X_{12}. \quad (11)$$

On the other hand, the equations (6)–(9), that in the following we shall call the *WXZ system*, are constant version of the spectral dependent Yang–Baxter systems for nonultralocal models presented in [14].

There are two natural questions related to the WXZ system:

- Is there a matrix X such that for any pair of matrices W, Z that solve the Yang–Baxter equations the triple (W, X, Z) solves the WXZ system?
- Is there for any matrix X a pair of matrices W, Z such that the triple (W, X, Z) solves the WXZ system?

Answers to both these questions are positive because the following two simple propositions hold

- Let W, Z are arbitrary solutions of the Yang–Baxter equations $[W, W, W] = 0$, $[Z, Z, Z] = 0$. Then the triple $(W, X = \mathbf{1}, Z)$ is a solution of the system (6)–(9).
- Let X is an arbitrary matrix $N^2 \times N^2$ and P is the permutation matrix. Then the triple $(W = P, X, Z = P)$ is a solution of the system (6)–(9).

Other simple solution of the WXZ system is $(W = R, X = R, Z = R)$, where R is an arbitrary solution of the Yang–Baxter equation.

Besides the above mentioned, one can find solutions of the WXZ system from the knowledge of solutions of the system (4)–(5). Namely, if (Q, R) is a solution of the system of Yang–Baxter type equations (4)–(5) then $(W = Q, X = R, Z = Q)$ and $(W = Q, X = R, Z = R^+QR^-)$ are solutions of the system (6)–(9).

Solution of the system is essentially facilitated by knowledge of its symmetries. It is easy to check that the set of solutions is invariant under both continuous transformations

$$\begin{aligned} W' &= \omega(T \otimes T)W(T \otimes T)^{-1} \\ X' &= \xi(T \otimes S)X(T \otimes S)^{-1} \\ Z' &= \zeta(S \otimes S)Z(S \otimes S)^{-1} \end{aligned} \tag{12}$$

where

$$\omega, \xi, \zeta \in \mathbf{C}, \quad T, S \in \mathcal{GL}(N, \mathbf{C}),$$

and discrete transformations

$$(W', X', Z') = (W^T, X^T, Z^T) \tag{13}$$

$$(W', X', Z') = (W^a, X, Z^b), \quad a = id, \#, \quad b = id, \# \tag{14}$$

$$(W', X', Z') = (W^c, X^-, Z^d), \quad c = +, -, \quad d = +, - \tag{15}$$

$$(W', X', Z') = (Z^c, X^+, W^d), \quad c = +, -, \quad d = +, - \tag{16}$$

where Y^T is transpose of Y , $Y^+ := PYP$, P being the permutation matrix, $Y^- := Y^{-1}$, $Y^{id} := Y$, $Y^\# := (Y^+)^- = (Y^-)^+$.

These symmetries can be of course composed, so that one can define e.g. the "antidiagonal transposition"

$$Y^{at} := (\sigma \otimes \sigma) Y^T (\sigma \otimes \sigma) \text{ where } \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (17)$$

and the symmetry

$$(W', X', Z') = (W^{at}, X^{at}, Z^{at}) \quad (18)$$

Beside that there are useful "conditional symmetries" that can be expressed as

Lemma 1: Let W, X, Z are matrices $N^2 \times N^2$ that solve WXZ system (6)–(9) and T, A are matrices $N \times N$ such that

$$[W, T \otimes \mathbf{1}] = 0, \quad (19)$$

$$X(T \otimes \mathbf{1}) = (A \otimes \mathbf{1})X \text{ or } (T \otimes \mathbf{1})X = X(A \otimes \mathbf{1}). \quad (20)$$

Then

$$W' = W, \quad X' = (T \otimes \mathbf{1})X, \quad Z' = Z \quad (21)$$

$$W' = W, \quad X' = X(T \otimes \mathbf{1}), \quad Z' = Z \quad (22)$$

and

$$W' = (T \otimes \mathbf{1})W(T \otimes \mathbf{1})^{-1}, \quad X' = X, \quad Z' = Z \quad (23)$$

solve the WXZ system as well.

Lemma 2: Let W, X, Z are matrices $N^2 \times N^2$ that solve WXZ system (6)–(9) and S, A are matrices $N \times N$ such that

$$[Z, S \otimes \mathbf{1}] = 0, \quad (24)$$

$$X(S \otimes \mathbf{1}) = (A \otimes \mathbf{1})X \text{ or } (S \otimes \mathbf{1})X = X(A \otimes \mathbf{1}). \quad (25)$$

Then

$$W' = W, \quad X' = (S \otimes \mathbf{1})X, \quad Z' = Z \quad (26)$$

$$W' = W, \quad X' = X(S \otimes \mathbf{1}), \quad Z' = Z \quad (27)$$

and

$$W' = W, \quad X' = X, \quad Z' = (S \otimes \mathbf{1})Z(S \otimes \mathbf{1})^{-1} \quad (28)$$

solve the WXZ system as well.

Proofs of these lemmas can be done by direct calculations. There are other symmetries of the WXZ system that are extensions of the twisting transformations of the solutions Yang–Baxter equation but we are not going to use them in the following as it seems that they do not produce equivalent quantum doubles.

3 Solution of the WXZ system in the dimension two

Even for the lowest nontrivial dimension two, solution of the system (6)–(9) represents a tremendous task, namely solving 256 cubic equations for 48 unknowns. That’s why it is understandable that the assistance of computer programs for symbolic calculations is essential in the following¹. On the other hand it does not mean that one can find the solutions by pure brute force, namely applying a procedure SOLVE to the system of the 256 equations. Moreover, the interpretation of the result would be extremely difficult as hundreds of solutions would appear, many of them related by the symmetries.

There are several strategies for solution of the problem given above. All of them are based on the symmetries of the system and the knowledge of the complete set of solutions of the Yang–Baxter equation in the dimension two [6].

One possible (and obvious) strategy is solving the equations (8),(9) for all pairs of matrices W, Z in the Hietarinta’s list of solutions of the Yang–Baxter equation. By this way we reduce the problem to 128 quadratic equations for 16-22 unknowns (depending on the number of parameters in the solutions of the Yang–Baxter equation).

Another strategy is to use the symmetry (12) to simplify the matrix X as much as possible, then solve the (linear in W or Z) equations (8),(9) and finally solve the Yang–Baxter equations (6),(7) for W and Z . This is an analogue of the strategy accepted in [11] for solving the system (4)–(5).

Both the mentioned strategies yield sets of equations and unknowns that are still too large to be solved by the computer program. The strategy that seems to work is a combination of the two previous.

First we find the matrices X that solve the equation (8) for each invertible W that belongs to the Hietarinta’s list. In the next step we solve the equation (9)

and finally determine Z from (7) using the results of the previous step. As the list of solutions is quite large it is essential in each step to factorize its results by the symmetries (12)–(16).

Detailed description of this procedure and results are presented in the following subsections.

3.1 Solutions of the Yang–Baxter equations

As it was mentioned, the important point is the knowledge of all invertible solutions of the constant Yang–Baxter equation $[W, W, W] = 0$. Their list up to the symmetries, denoted \mathcal{S} , consists of twelve (intersecting) subsets [6, 2, 3] parametrized by up to three complex numbers.

$$\mathcal{S} := \{R_{3.1}(r, s, t), R_{2.1}(r, s), R_{2.2}(r, s), R_{2.3}(u, v), R_{1.1}(r), R_{1.2}(s), R_{1.3}(u),$$

¹Reduce 3.6 and Maple V was used

$$R_{1.4}(t), R_{0.1}, R_{0.2}, R_{0.3}, R_{0.4} \mid r, s, t \in \mathbf{C} \setminus \{0\}, u, v \in \mathbf{C} \quad (29)$$

The matrices R are defined in the Appendix 5.1.

For later use we shall define three subsets of \mathcal{S} , namely

$$\mathcal{S}_{5V} := \{R_{2.1}(r, s), R_{2.2}(r, s), R_{3.1}(r, s, t) \mid r, s, t \in \mathbf{C} \setminus \{0\}\}, \quad (30)$$

$$\mathcal{S}_{8V} := \{R_{3.1}(r, s, t), R_{2.1}(r, s), R_{2.2}(r, s), R_{1.1}(r), R_{1.2}(s), R_{1.4}(t), \\ R_{0.1}, R_{0.2}, R_{0.3}, R_{0.4} \mid r, s, t \in \mathbf{C} \setminus \{0\}\} \quad (31)$$

$$\mathcal{S}_{ST} := \{R_{3.1}(1, 1, 1), R_{2.3}(u, v), R_{1.3}(u), R_{0.1} \mid u, v \in \mathbf{C}\}. \quad (32)$$

They are the sets of solutions of the "five-vertex" form

$$R_{5V} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & q & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \quad (33)$$

"eight-vertex" form

$$R_{8V} = \begin{pmatrix} a & 0 & 0 & x \\ 0 & b & p & 0 \\ 0 & q & c & 0 \\ y & 0 & 0 & d \end{pmatrix}, \quad (34)$$

and "special triangular" form

$$R_{ST} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & d & e & 1 \end{pmatrix}. \quad (35)$$

It is easy to check that

$$\mathcal{S} = \mathcal{S}_{8V} \cup \mathcal{S}_{ST}, \quad \mathcal{S}_{5V} \subset \mathcal{S}_{8V} \quad (36)$$

The reason for this partition is that these subsets have common symmetries and common matrices X that solve the equation (8). Namely, each element of \mathcal{S}_{5V} is invariant under

$$R' = (A \otimes A)R(A \otimes A)^{-1} \quad (37)$$

where

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad (38)$$

while each element of \mathcal{S}_{8V} is invariant under (37) where

$$A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \quad (39)$$

and each element of \mathcal{S}_{ST} is invariant under (37) where

$$A = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}. \quad (40)$$

These invariances imply the existence of solutions $X = A \otimes B$ of the equation (8) common to all elements of the subsets \mathcal{S}_{5V} , \mathcal{S}_{8V} or \mathcal{S}_{ST} . Actually we shall find that slightly more general solutions are generic for each of the subsets.

3.2 Solutions of the WXX equation

The goal of this subsection is to find all invertible matrices X that solve the equation

$$[W, X, X] = 0 \quad (41)$$

for all invertible solutions W of the Yang–Baxter equation (6). First of all we can exploit the symmetry (12) to reduce the matrix X to one of the forms A_1, \dots, A_{14} given in the Appendix 5.2. Then we choose a solution W of the Yang–Baxter equation and solve X from the equation (41). The list of the solutions up to the WXZ symmetries (12), (14),(16), (21), (22) can be described in terms of 23 classes of matrices parametrized by complex numbers and given in the Appendix 5.3.

As mentioned in the previous subsection, there are solutions common to all elements in subsets \mathcal{S}_{5V} , \mathcal{S}_{8V} or \mathcal{S}_{ST} . Namely:

- If $W \in \mathcal{S}_{5V}$ then $X_1(a, b, c)$ and $X_2(a, b, c)$ solve (41) for all $a, b, c \in \mathbf{C}$.
- If $W \in \mathcal{S}_{8V}$ then $X_1(a, \epsilon_1, \epsilon_2 a)$, and $X_2(\epsilon_1, a, \epsilon_1 a)$ solve (41) for all $a \in \mathbf{C}$ and $\epsilon_1, \epsilon_2 = \pm 1$.
- If $W \in \mathcal{S}_{ST}$ then $X_3(a, b, c)$ and $X_4(a, b, c)$ solve (41) for all $a, b, c \in \mathbf{C}$.

We shall call these solutions *generic*. Notation of the X-matrices above as well as below corresponds to that in the Appendix 5.3.

Below we display all invertible non-generic solutions of (41). The list is complete up to the WXZ symmetries (12), (14),(16), (21), (22). That's why we also present the transformations that leave given W invariant or form invariant, i.e. invariant up to the change of parameters.

The ranges of parameters are

$$r, s, t \in \mathbf{C} \setminus \{0\}, \quad u, v, a, b, c, d, e, f, g, h \in \mathbf{C}, \quad \epsilon, \epsilon_1, \epsilon_2 = \pm 1.$$

List of non-generic solutions of the equation (41):

- $W = R_{3.1}(r, s, t)$.

This matrix is invariant w.r.t. (37),(38) and

$$R_{3.1}(r, s, t)^T = R_{3.1}(r, s, t), \quad R_{3.1}(r, s, t)^{-1} = R_{3.1}(r^{-1}, s^{-1}, t^{-1}), \quad (42)$$

$$R_{3.1}(r, s, t)^+ = R_{3.1}(s, r, t) \quad (43)$$

$$(T \otimes T)R_{3.1}(r, s, t)(T \otimes T)^{-1} = t R_{3.1}\left(\frac{s}{t}, \frac{r}{t}, \frac{1}{t}\right), \quad \text{where } T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (44)$$

The non-generic solutions of the equation $[W, X, X] = 0$ up to the symmetries exist only for special values of the parameters r, s, t .

1. $W = R_{3.1}(s, s, 1), \quad X = X_5(a, b, c)$
2. $W = R_{3.1}(s, s, 1), \quad X = X_6(a, b, c)$
3. $W = R_{3.1}(s, s, 1), \quad X = X_7(a, b)$
4. $W = R_{3.1}(s, -s, 1), \quad X = X_8(a)$
5. $W = R_{3.1}(1/s, s, \epsilon), \quad X = X_9(s, a, \epsilon sa)$
6. $W = R_{3.1}(\epsilon, \epsilon, -\epsilon), \quad X = X_{10}(\epsilon, b, -b, c)$
7. $W = R_{3.1}(-1, -1, 1), \quad X = X_{11}(a, b, c, d)$
8. $W = R_{3.1}(1, 1, 1), \quad X = X_{22}(a, b, c, d, e, f, g, h)$
9. $W = R_{3.1}(1, 1, 1), \quad X = X_{23}(a, b, c, d, e, f, g, h)$

- $W = R_{2.1}(r, s)$

This matrix is invariant w.r.t. (37),(38) and

$$R_{2.1}^{at}(r, s) = R_{2.1}(s, r), \quad R_{2.1}(r, s)^{-1} = R_{2.1}(r^{-1}, s^{-1}) \quad (45)$$

$$(T \otimes T)R_{2.1}^+(r, s)(T \otimes T)^{-1} = R_{2.1}(r, s), \quad \text{where } T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (46)$$

The non-generic solutions of the equation $[W, X, X] = 0$ up to the WXZ symmetries are

1. $W = R_{2.1}(r, s), \quad X = X_9(r^{-1}, a, sa)$
2. $W = R_{2.1}(s, s), \quad X = X_{10}(s^{-1}, a, sa, b)$
3. $W = R_{2.1}(1, -1), \quad X = X_{12}(a, b)$
4. $W = R_{2.1}(i, i), \quad X = X_{13}(a)$
5. $W = R_{2.1}(i, -i), \quad X = X_8(a)$ (coincide with the case 4 in $W = R_{3.1}$).
6. $W = R_{2.1}(\epsilon, \epsilon)$, coincide with the cases 1-3,5,7-9 in $W = R_{3.1}$.

- $W = R_{2.2}(r, s)$.

This matrix is invariant w.r.t. (37),(38) and

$$R_{2.2}^{at}(r, s) = -rsR_{2.2}(-r^{-1}, -s^{-1}), \quad R_{2.2}(r, s)^{-1} = R_{2.2}(r^{-1}, s^{-1}) \quad (47)$$

$$(T \otimes T)R_{2.2}^+(r, s)(T \otimes T)^{-1} = -rsR_{2.2}(-s^{-1}, -r^{-1}), \quad \text{where } T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (48)$$

The non-generic solutions of the equation $[W, X, X] = 0$ up to the WXZ symmetries are

1. $W = R_{2.2}(r, s), \quad X = X_9(r^{-1}, a, -ar^{-1})$
2. $W = R_{2.2}(s, s), \quad X = X_{10}(s^{-1}, a, -s^{-1}a, g)$
3. $W = R_{2.2}(1, -1), \quad R_{2.2}(i, i)$ coincide with cases 3,4 in $W = R_{2.1}$.

- $W = R_{2.3}(u, v)$

This matrix is invariant w.r.t. (37), (40) and

$$R_{2.3}^{at}(u, v) = R_{2.3}(u, v) \quad (49)$$

$$(T \otimes T)R_{2.3}^{-1}(u, v)(T \otimes T)^{-1} = R_{2.3}(u, 2u - v), \quad \text{where } T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (50)$$

$$(T \otimes T)R_{2.3}^+(u, v)(T \otimes T)^{-1} = R_{2.3}(1/u, v/u^2), \quad \text{where } T = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \quad (51)$$

and there are no non-generic solutions of the equation $[W, X, X] = 0$ up to the WXZ symmetries.

- $W = R_{1.1}(r)$.

This matrix is invariant w.r.t. (37),(39) and

$$R_{1.1}^t(r) = R_{1.1}(r), \quad R_{1.1}(r)^+ = R_{1.1}(r) \quad (52)$$

$$(T \otimes T)R_{1.1}^{-1}(r)(T \otimes T)^{-1} = -\frac{1}{4}R_{1.1}(-r), \quad \text{where } T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \quad (53)$$

The non-generic solutions of the equation $[W, X, X] = 0$ up to the WXZ symmetries are

1. $W = R_{1.1}(r), \quad X = X_{10}(r, -\epsilon, \epsilon r, \epsilon)$
2. $W = R_{1.1}(r), \quad X = X_{14}(a, \epsilon, \frac{r+1}{r-1}), \quad r \neq 0, \pm 1$
3. $W = R_{1.1}(i), \quad X = X_{15}(a)$

- $W = R_{1.2}(s)$

This matrix is invariant w.r.t. (37), (39) and

$$(T \otimes T)R_{1.2}^{-1}(s)(T \otimes T)^{-1} = R_{1.2}(1/s), \text{ where } T = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{s} \end{pmatrix} \quad (54)$$

and there are the following non-generic solutions of the equation $[W, X, X] = 0$ up to the symmetries:

1. $W = R_{1.2}(s), X = X_9(1, \epsilon, -\epsilon)$
2. $W = R_{1.2}(s), X = X_{10}(s^{-1}, \epsilon, -\epsilon s^{-1}, \frac{\epsilon}{s-1})$,
3. $W = R_{1.2}(s), X = X_{12}(a, b), a^2 - b^2 = s + 1$

- $W = R_{1.3}(u)$

This matrix is invariant w.r.t. (37), (40) and

$$R_{1.3}^+(u) = R_{1.3}^{-1}(u) = (T \otimes T)R_{1.3}(u)(T \otimes T)^{-1}, \text{ where } T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (55)$$

and there are the following non-generic solutions of the equation $[W, X, X] = 0$ up to the symmetries:

1. $W = R_{1.3}(u), X = X_{16}(a, b, b - u - 1, -ua), u \neq -1$
2. $W = R_{1.3}(1), X = X_{17}(a, b)$

- $W = R_{1.4}(t)$.

This matrix is invariant w.r.t. (37), (38) and

$$R_{1.4}^t(t) = R_{1.4}(t), R_{1.4}^+ = R_{1.4}(t), R_{1.4}^{-1}(t) = R_{1.4}(t^{-1}), \quad (56)$$

$$(T \otimes T)R_{1.4}(t)(T \otimes T)^{-1} = R_{1.4}(t), \text{ where } T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (57)$$

The non-generic solutions of the equation $[W, X, X] = 0$ up to the symmetries are

1. $W = R_{1.4}(t), X = X_5(a, \epsilon, a)$
2. $W = R_{1.4}(t), X = X_6(a, 1, -a)$
3. $W = R_{1.4}(t), X = X_{15}(a)$
4. $W = R_{1.4}(t), X = X_{18}(a)$
5. $W = R_{1.4}(\pm 1)$. These matrices can be transformed to $R_{3.1}(-1, -1, 1)$ by symmetry transformations so that these solutions are equivalent to the cases 1, 2, 3, 5, 7, 8 ($s = -1, \epsilon = -1$) of $W = R_{3.1}$.

- $W = R_{0.1}$

This matrix is invariant w.r.t. (37), (39) and

$$R_{0.1}^{at} = R_{0.1}, \quad R_{0.1}^+ = R_{0.1}, \quad (58)$$

$$(T \otimes T)R_{0.1}^{-1}(T \otimes T)^{-1} = R_{0.1}, \quad \text{where } T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad (59)$$

and there are the following non-generic solutions of the equation $[W, X, X] = 0$ up to the symmetries:

1. $W = R_{0.1}, \quad X = X_{19}(a, b, c, \epsilon)$
2. $W = R_{0.1}, \quad X = X_{20}(a, b, c)$

- $W = R_{0.2}$

This matrix is invariant w.r.t. (37), (39) and

$$R_{0.2}^{at} = R_{0.2}, \quad R_{0.2}^+ = R_{0.2}, \quad (60)$$

$$(T \otimes T)R_{0.2}^{-1}(T \otimes T)^{-1} = R_{0.2}, \quad \text{where } T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad (61)$$

and there are the following non-generic solutions of the equation $[W, X, X] = 0$ up to the symmetries:

1. $W = R_{0.2}, \quad X = X_9(-1, \epsilon, -\epsilon)$
2. $W = R_{0.2}, \quad X = X_{11}(\epsilon, b, 0, 1)$

- $W = R_{0.3}$

This matrix is invariant w.r.t. (37), (39) and

$$R_{0.3}^T = R_{0.3}, \quad (T \otimes T)R_{0.3}^+(T \otimes T)^{-1} = R_{0.3}, \quad \text{where } T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (62)$$

$$2(T \otimes T)R_{0.3}^{-1}(T \otimes T)^{-1} = R_{0.3}, \quad \text{where } T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad (63)$$

and there are the following non-generic solutions of the equation $[W, X, X] = 0$ up to the symmetries:

1. $W = R_{0.3}, \quad X = X_{10}(a, -1, a, i)$
2. $W = R_{0.3}, \quad X = X_{21}(a, b)$

- $W = R_{0.4} = P$

Arbitrary matrix X solves the equation $[W, X, X] = 0$.

3.3 Solution of XXZ and ZZZ equations for given X

In the preceding subsection we have found all pairs (W, X) of invertible 4×4 matrices that solve the equations (6), (8). The solutions are given in terms of the matrices X presented in the Appendix 5.3. The last step to do for solving the WXZ system is to find the invertible 4×4 matrices Z that for given X solve the equations

$$[X, X, Z] = 0, [Z, Z, Z] = 0. \quad (64)$$

Trivial solution for any X is $Z = R_{0.4} = P =$ permutation matrix. Below we display the list of nontrivial solutions of the equations (64) for all X -matrices from the subsection 5.3.

Let us note that in spite of the fact that the matrices Z must solve the Yang–Baxter equation, they need not belong to the list of solutions in the Appendix 5.1 because in general we have not at disposal all the transformations (69) up to which that list is complete, because they might already be used as a part of WXZ symmetries (12), (14), (16) to fix the matrices W and X . The only symmetry that we can use in general is

$$(W', X', Z') = (W, X, Z^\#). \quad (65)$$

Beside this we can use the WXZ symmetries that leave X (and corresponding W) invariant. For some matrices X the whole sets of the solutions of the Yang–Baxter equation \mathcal{S} or \mathcal{S}_{5V} or \mathcal{S}_{8V} or \mathcal{S}_{ST} solve the equations (64). Due to the restricted symmetries of matrices Z

it is convenient for classification of solutions to introduce another special set of Yang–Baxter solutions

$$\begin{aligned} \mathcal{R}_{8V} := & \{R_{3.1}(r, s, t), R_{2.1}(r, s), R_{2.2}(r, s), QR_{1.1}(r)Q^{-1}, \\ & QR_{1.2}(s)Q^{-1}, QR_{1.2}(s)^T Q^{-1}, QR_{1.4}(t)Q^{-1}, \\ & QR_{0.1}Q^{-1}, QR_{0.2}Q^{-1}, QR_{0.3}Q^{-1}, QR_{0.2}^T Q^{-1}, QR_{0.3}^T Q^{-1}, \\ & \left(\begin{array}{cccc} 1 & 0 & 0 & p/q^2 \\ 0 & 1 & p & 0 \\ 0 & p & 1 & 0 \\ pq^2 & 0 & 0 & 1 \end{array} \right), \left(\begin{array}{cccc} 1 & 0 & 0 & p/q^2 \\ 0 & -1 & p & 0 \\ 0 & p & -1 & 0 \\ pq^2 & 0 & 0 & 1 \end{array} \right) | \\ & q, r, s, t \in \mathbf{C} \setminus \{0\}, p \in \mathbf{C} \setminus \{1, -1\} \} \end{aligned} \quad (66)$$

where

$$Q = D(q) = \text{diag}(1, q, q, q^2) \quad (67)$$

This is the set of the "eight-vertex" solutions up to the symmetries $R' = R^\#$. The last two matrices can be transformed to $R_{3.1}(\frac{1-p}{1+p}, \frac{1-p}{1+p}, 1)$ and $R_{1.4}(\frac{1+p}{1-p})$ by similarity transformations.

Below we display nontrivial invertible solutions of (64). The list is complete up to the symmetries (12)–(16), (26), (27). The ranges of parameters are

$$q, r, s, t, x \in \mathbf{C} \setminus \{0\}, \quad u, v, a, b, c, d, e, f, g, h \in \mathbf{C}, \quad \epsilon, \epsilon_1, \epsilon_2 = \pm 1,$$

and $\mathcal{S}, \mathcal{S}_{8V}, \mathcal{S}_{5V}, \mathcal{S}_{ST}$ are defined by (29)–(32).

List of nontrivial solutions of the equation (64):

- $X = X_1(a, b, c)$

The nontrivial solutions for general values of a, b, c are

1. $Z \in \mathcal{S}_{5V}$

For special values of a, b, c we get the following solutions

2. $(a = -1 \wedge c = \pm b)$ or $(a = 1 \wedge c = -b) : Z \in \mathcal{R}_{8V}$
3. $a = 1 \wedge c = b : Z \in \mathcal{S} \cup \{R_{1.2}^T(s)\}$.

- $X = X_2(a, b, c)$

The nontrivial solutions for general values of a, b, c are

1. $Z \in \mathcal{S}_{ST}$

There are no other solutions for special values of a, b, c .

- $X = X_3(a, b, c)$

The nontrivial solutions for general values of a, b, c are

1. $Z \in \mathcal{S}_{5V}$

For special values of a, b we get the following solutions

2. $a = -1 \wedge b = -c : Z \in \mathcal{R}_{8V}$
3. $a = 1 \wedge c = b : Z \in \mathcal{S} \cup \{R_{1.2}^T(s)\}$

- $X = X_4(a, b, c)$

The nontrivial solutions for general values of a, b, c are

1. $Z \in \mathcal{S}_{ST}$

There are no other solutions for special values of a, b, c .

- $X = X_5(a, b, c)$

There are no nontrivial solutions for general values of a, b, c . For special values we get

1. $b = -1, a = 1/c : Z = R_{1.1}(i)$

2. $b = -1, a = 1/c : Z = R_{1.4}(x)$
 3. $b = 1 : Z \in \mathcal{S}_{5V}$
 4. $b = 1 \wedge a = 1/c : Z \in \mathcal{R}_{8V}$
- $X = X_6(a, b, c)$
There are no nontrivial solutions for general values of a, b, c . For special values we get
 1. $a = b/c : Z = R_{1.1}(i)$
 2. $a = b/c : Z = R_{1.4}(x)$
 3. $a = c/b : Z \in \mathcal{S}_{5V}$
 4. $a = -1, b = -c : Z \in \mathcal{R}_{8V}$
 5. $a = 1, b = c : Z \in \mathcal{S} \cup \{R_{1.2}^T(s)\}$
 - $X = X_7(a, b)$
There are no nontrivial solutions for general values of a, b . For special values we get
 1. $a = b : Z \in \mathcal{S}_{ST}$
 2. $a = -b : Z = R_{3.1}(-1, -1, 1)$
 3. $a = -b : Z = (S(x, y) \otimes S(x, y))^{-1} R_{0.2}(S(x, y) \otimes S(x, y))$
where $S(x, y) = \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix}$
 - $X = X_8(a)$
The nontrivial solutions for general values of a are
 1. $Z = (A \otimes A) R_{1.4}(x) (A \otimes A)^{-1}$
 2. $Z = (A \otimes A) R_{0.3}(A \otimes A)^{-1}$ where $A = \begin{pmatrix} \frac{1+\epsilon i}{\sqrt{2}} \sqrt{a} & \epsilon \sqrt{a} \\ \frac{1-\epsilon i}{\sqrt{2}} & i \end{pmatrix}$
 3. $Z = (B \otimes B) R_{3.1}(x, -x, 1) (B \otimes B)^{-1}$, where $B = \begin{pmatrix} -\sqrt{ai} & \sqrt{ai} \\ 1 & 1 \end{pmatrix}$

There are no other solutions for special values of a .
 - $X = X_9(b, c, d)$
The nontrivial solutions for general values of b, c, d are
 1. $Z = R_{2.1}(c, b/d)$
For special values of b, c, d we get the following solutions ($D(q)$ is given by (67)).
 2. $b = c = 1 = -d : Z = D(q) R_{1.2}(x) D(q)^{-1}$

3. $b = d = -c = 1$: $Z = D(q)R_{1.2}^{at}(x)D(q)^{-1}$
 4. $c = -d/b$: $Z = R_{2.2}(c, x)$
 5. $c = -1 \wedge b = -d = \epsilon$: $Z = D(q)R_{0.2}D(q)^{-1}$
 6. $b = 1 \wedge c = d$: $Z = (S(x) \otimes S(x))R_{3.1}(d, 1/d, 1)(S(x) \otimes S(x))^{-1}$
where $S(x) = \begin{pmatrix} 1-d & 0 \\ x & x \end{pmatrix}$, $d \neq 1$
 7. $b = c = d = 1$: $Z \in \mathcal{S}_{ST}$
 8. $c = 1 \wedge b = d = \epsilon$: $Z = R_{0.1}$
- $X = X_{10}(b, c, d, g)$
There are no nontrivial solutions for general values of parameters. For special values we get the following solutions ($D(q)$ is given by (67)).
 1. $b = \epsilon d$, $c = \epsilon$: $Z = R_{3.1}(\epsilon, \epsilon, 1)$
 2. $b = \epsilon d$, $c = -\epsilon$: $Z = R_{2.1}(-\epsilon, \epsilon)$
 3. $b = -d$, $c = 1$: $Z = D(\sqrt{gx})R_{1.2}(-1 + x(1 - b^2))D(\sqrt{gx})^{-1}$
 4. $b = d$, $c = -1$: $Z = D(\sqrt{gx})R_{1.2}^{at}(-1 - x(1 - b^2))D(\sqrt{gx})^{-1}$
 5. $b = c = d = 1$, : $Z \in \mathcal{S}_{ST}$
 6. $b = 1$, $c = d = -1$: $Z = Z = (A(x, y) \otimes A(x, y))^{-1}R_{0.2}(A(x, y) \otimes A(x, y))$
where $A(x, y) = \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix}$
 7. $d = -b = -c = 1$: $Z = R_{0.2}$
 8. $c = -b = -d = 1$: $Z = R_{0.1}$
 - $X = X_{11}(a, b, c, d)$, $c \neq 0$
There are no nontrivial solutions for general values of parameters. For special values we get
 1. $a = c$: $Z = (A \otimes A)R_{3.1}(x, x, 1)(A \otimes A)^{-1}$
 2. $a = -c$: $Z = (A \otimes A)R_{1.4}(x)(A \otimes A)^{-1}$
where $A = \begin{pmatrix} \sqrt{b} & \sqrt{b} \\ -i & i \end{pmatrix}$
 3. $a = c$, $d = b$: $Z = (B(y) \otimes B(y))R_{3.1}(x, x, 1)(B(y) \otimes B(y))^{-1}$
where $B(y) = \begin{pmatrix} \sqrt{b}(1 + \epsilon\sqrt{2}) & y\sqrt{b}(1 - \epsilon\sqrt{2}) \\ 1 & y \end{pmatrix}$
 4. $a = c$, $d = b$: $Z = (C \otimes C)R_{1.4}(x)(C \otimes C)^{-1}$
where $C = \begin{pmatrix} \epsilon_1\sqrt{b}(\sqrt{2} + \epsilon_2) & \sqrt{b} \\ -\epsilon_1\epsilon_2 & 1 + \epsilon_2\sqrt{2} \end{pmatrix}$

5. $a = c, d = b$: $Z = (D(x) \otimes D(x))R_{1.4}(1)(D(x) \otimes D(x))^{-1}$
where $D(x) = \begin{pmatrix} x\sqrt{b} & \sqrt{b} \\ \epsilon\sqrt{2} - x & \epsilon x\sqrt{2} - 1 \end{pmatrix}$
6. $a = c, d = b$: $Z = (E \otimes E)R_{1.4}(x)(E \otimes E)^{-1}$
where $E = \begin{pmatrix} i\sqrt{b}\epsilon_1 & i\sqrt{ib}\epsilon_2 \\ 1 & -\epsilon_1\epsilon_2\sqrt{i} \end{pmatrix}$
7. $a = c, d = b$: $Z = (F(y) \otimes F(y))R_{1.4}(1)(F(y) \otimes F(y))^{-1}$
where $F(y) = \begin{pmatrix} \sqrt{b}(\epsilon_1\sqrt{2} + y\epsilon_2) & \sqrt{b} \\ y & -\epsilon_1 y\sqrt{2} - \epsilon_2 \end{pmatrix}$
8. $a = -c, d = -b$: $Z = (G(y) \otimes G(y))R_{3.1}(x, -x, 1)(G(y) \otimes G(y))^{-1}$
where $G(y) = \begin{pmatrix} \epsilon\sqrt{d} & -y\epsilon\sqrt{d} \\ 1 & y \end{pmatrix}$
9. $a = -c, d = -b$: $Z = (H \otimes H)R_{0.3}(H \otimes H)^{-1}$
where $H = \begin{pmatrix} i\sqrt{b} & i\epsilon\sqrt{b} \\ 1 & -\epsilon \end{pmatrix}$

- $X = X_{12}(a, b)$

The nontrivial solutions for general values of a, b are

1. $Z = (A \otimes A)R_{2.2}(b/a, b/a)(A \otimes A)^{-1}$
2. $Z = (A \otimes A)R_{1.1}(a/b)(A \otimes A)^{-1}$ where $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
3. $Z = (B \otimes B)R_{1.2}(b/a)(B \otimes B)^{-1}$
where $B = \begin{pmatrix} 1 & q \\ 1 & -q \end{pmatrix}, q^2 = 1 - b/a$
4. $Z = (C \otimes C)R_{1.2}(-b/a)(C \otimes C)^{-1}$
where $C = \begin{pmatrix} 1 & q \\ -1 & q \end{pmatrix}, q^2 = 1 + b/a$
5. $Z = (D \otimes D)R_{1.2}^T(-a/b)(D \otimes D)^{-1}$
where $D = \begin{pmatrix} q & 1 \\ -q & 1 \end{pmatrix}, q^2 = -1 - a/b$
6. $Z = (E \otimes E)R_{1.2}^T(a/b)(E \otimes E)^{-1}$
where $E = \begin{pmatrix} q & 1 \\ q & -1 \end{pmatrix}, q^2 = -1 + a/b$
7. $Z = (F \otimes F)R_{0.3}(F \otimes F)^{-1}$ where $F = \begin{pmatrix} \sqrt{i} & 1 \\ \sqrt{ib} & -\sqrt{b} \end{pmatrix}$

There are no other solutions for special values of a, b .

- $X = X_{13}(g)$

There is no nontrivial solution of the equations (64).

- $X = X_{14}(a, \epsilon, \frac{r+1}{r-1})$

The nontrivial solutions for general values of a , $r \neq 0, \pm 1$ are

1. $Z = (A \otimes A)R_{1.1}(\epsilon a/c)(A \otimes A)$ where $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $c^2 = a^2 - \frac{r-1}{r+1}$

- $X = X_{15}(a)$

The nontrivial solutions for general values of a are

1. $Z = (A \otimes A)R_{3.1}(x, x, 1)(A \otimes A)^{-1}$

where $A = \begin{pmatrix} -q & q \\ 1 & 1 \end{pmatrix}$, $q^2 = a^2 - 1$

For special values of a we get the following solutions

2. $a = \pm 1$: $Z = R_{0.1}$

3. $a = 0$: $Z = (B \otimes B)R_{3.1}(x, x, 1)(B \otimes B)^{-1}$

4. $a = 0$: $Z = (B \otimes B)R_{1.4}(x)(B \otimes B)^{-1}$ where $B = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

5. $a = 0$: $Z = (C \otimes C)R_{1.4}(x)(C \otimes C)^{-1}$ where $C = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$

- $X = X_{16}(a, b, c, d)$

There are no nontrivial solutions for general values of parameters. For special values we get

1. $a = d = 0$: $Z \in \mathcal{S}_{5V}$

2. $c = b(1 + a - d)$: $Z = R_{1.3}(d - a - 1)$

- $X = X_{17}(a, b)$

There is no nontrivial solution of the equations (64).

- $X = X_{18}(b)$

The nontrivial solutions for $b \neq 0, \epsilon$ are

1. $Z = (A \otimes A)R_{1.4}(x)(A \otimes A)^{-1}$,

where $A = \begin{pmatrix} i & -ip \\ q & pq \end{pmatrix}$, $q^2 = 1 - b^2$, $p^4 = (b - iq)^2$.

For special values of b we get

2. $b = \epsilon$: $Z = (B(x) \otimes B(x))R_{0.2}(B(x) \otimes B(x))^{-1}$

where $B(x) = \begin{pmatrix} -1 & x \\ 4\epsilon & 0 \end{pmatrix}$

3. $b = 0$ see $X_8(a = 1)$

- $X = X_{19}(a, b, c, \epsilon)$

There are no nontrivial solutions for general values of parameters. For special values we get

1. $\epsilon = 1, b = 0$: $Z \in \mathcal{S}_{5V}$
2. $\epsilon = -1, a = \pm 1$: $Z = R_{1.1}(i)$
3. $\epsilon = -1, a = \pm 1$: $Z = R_{1.4}(x)$
4. $\epsilon = -1, c = ab$: $Z \in \mathcal{S}_{5V}$
5. $\epsilon = -1, a = -1, c = -b$: $Z \in \mathcal{R}_{8V}$
6. $\epsilon = -1, a = 1, c = b$: $Z \in \mathcal{S} \cup \{R_{1.2}^T(s)\}$

- $X = X_{20}(a, b, c)$

There are no nontrivial solutions for general values of parameters. For special values we get

1. $b = 0$: $Z = R_{3.1}(-1, -1, 1)$
2. $b = 0$: $Z = (A(x, y) \otimes A(x, y))^{-1} R_{0.2}(A(x, y) \otimes A(x, y))$
where $A(x, y) = \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix}$
3. $a = cb$: $Z \in \mathcal{S}_{ST}$

- $X = X_{21}(a, b)$

There are no nontrivial solutions for general values of parameters. For special values we get

1. $b = 0$: $Z = (S \otimes S) R_{0.3}(S \otimes S)^{-1}$ where $S = \begin{pmatrix} 1 & \sqrt{i} \\ 1 & -\sqrt{i} \end{pmatrix}$
2. $a = b = 0$: $Z = (S \otimes S) R_{3.1}(x, -x, 1)(S \otimes S)^{-1}$ where $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$
3. $a = b = 0$: $Z = (S \otimes S) R_{1.4}(x)(S \otimes S)^{-1}$ where $S = \begin{pmatrix} 1 & \sqrt{i} \\ -1 & \sqrt{i} \end{pmatrix}$

- $X = X_{22}(a, b, c, d, e, f, g, h)$ and $X = X_{23}(a, b, c, d, e, f, g, h)$.

Solving the system (64) for these two matrices is rather difficult. We can, however, simplify the task by the following way:

Both the matrices solve the equation (41) only for $W = \mathbf{1}$, so that any Z that solve (64) for X_{22}, X_{23} gives solution $(\mathbf{1}, X, Z)$ of the WXZ system. This solution is equivalent to $(Z^\pm, X^+, \mathbf{1})$ due to (16). It is always possible to transform Z^\pm by WXZ symmetries to the form contained in \mathcal{S} . It means that for $Z \neq \mathbf{1}$ we can find the solution of the WXZ by symmetries investigating $W \neq \mathbf{1}$. It remains to investigate the case $Z = \mathbf{1}$ or, in other words, to solve the equation $[X, X, \mathbf{1}] = 0$ where X is of the form X_{22} or X_{23} . It turns out that the resulting matrices can be always transformed by the symmetry transformation $X' = (T \otimes \mathbf{1})X(T \otimes \mathbf{1})^{-1}$ to X_1, X_2, X_3, X_4 .

3.4 List of solutions of the WXZ system

Combining the results of the subsections 3.2 and 3.3 we get the list of solutions of the WXZ system (6)–(9) presented below. By construction the list is complete up to the symmetries (12)–(16) and (21),(23),(28).

Let us remind that $a, b, c, d, g, h \in \mathbf{C}$, $\epsilon, \epsilon_1, \epsilon_2, \epsilon_3 = \pm 1$, and $\mathcal{S}, \mathcal{S}_{8V}, \mathcal{S}_{5V}, \mathcal{S}_{ST}$ are defined by (29)–(32).

Generic solutions:

1. $W \in \mathcal{S}, \quad X = \mathbf{1}, \quad Z \in \mathcal{S} \cup \{R_{1,2}^T(s)\}$
2. $W \in \mathcal{S}_{8V}, \quad X = \text{diag}(1, 1, 1, -1), \quad Z \in \mathcal{S}_{8V} \cup \{R_{1,2}^T(s)\}$
3. $W \in \mathcal{S}_{8V}, \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Z = R_{3,1}(x, x, 1)$
4. $W \in \mathcal{S}_{8V}, \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Z = R_{1,4}(x)$
5. $W \in \mathcal{S}_{5V}, \quad X = \text{diag}(1, 1, 1, a), \quad Z \in \mathcal{S}_{5V} \text{ or } Z = P$
6. $W \in \mathcal{S}_{5V}, \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a & 1 \end{pmatrix}, \quad Z \in \mathcal{S}_{ST} \text{ or } Z = P$
7. $W \in \mathcal{S}_{ST}, \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & 0 & 0 & 1 \end{pmatrix}, \quad Z \in \mathcal{S}_{ST} \text{ or } Z = P$

Non-generic solutions:

8. $W = R_{3,1}(s, s, 1), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad Z = P$

$$9. \ W = R_{3.1}(s, s, 1), \ X = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}, \ Z = R_{3.1}(x, x, 1)$$

$$10. \ W = R_{3.1}(s, s, 1), \ X = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}, \ Z = R_{1.4}(x)$$

$$11. \ W = R_{3.1}(s, s, 1), \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ Z = R_{1.1}(i)$$

$$12. \ W = R_{3.1}(s, s, 1), \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ Z = R_{1.4}(x)$$

$$13. \ W = R_{3.1}(s, s, 1), \ X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ Z = P$$

$$14. \ W = R_{3.1}(s, s, 1), \ X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \\ 1 & 0 & 0 & 0 \\ 0 & 1/a & 0 & 0 \end{pmatrix}, \ Z = R_{1.1}(i)$$

$$15. \ W = R_{3.1}(s, s, 1), \ X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \\ 1 & 0 & 0 & 0 \\ 0 & 1/a & 0 & 0 \end{pmatrix}, \ Z = R_{1.4}(x)$$

$$16. \ W = R_{3.1}(s, s, 1), \ X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ c & 1 & 0 & 0 \end{pmatrix}, \ Z = P$$

$$17. \ W = R_{3.1}(s, s, 1), \ X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}, \ Z = R_{3.1}(-1, -1, 1)$$

$$18. W = R_{3.1}(s, s, 1), \quad X = \begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 1 & x \\ x & 0 & 0 & 0 \\ -1 & x & 0 & 0 \end{pmatrix}, \quad Z = R_{0.2}$$

$$19. W = R_{3.1}(s, -s, 1), \quad X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & a & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Z = P$$

$$20. W = R_{3.1}(s, -s, 1), \quad X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & q & 0 & 0 \\ -q & 0 & 0 & 0 \end{pmatrix}, \quad Z = R_{3.1}(x, -x, 1)$$

$$21. W = R_{3.1}(s, -s, 1), \quad X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i\epsilon & 0 \\ 0 & i\epsilon q & 0 & 0 \\ q & 0 & 0 & 0 \end{pmatrix}, \quad Z = R_{1.4}(x)$$

$$22. W = R_{3.1}(s, -s, 1), \quad X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i\epsilon & 0 \\ 0 & i\epsilon q & 0 & 0 \\ q & 0 & 0 & 0 \end{pmatrix}, \quad Z = R_{0.3}$$

$$23. W = R_{3.1}(\epsilon_1, \epsilon_1, 1), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon_1 & 1 & 0 \\ 0 & 0 & \epsilon_2 & 0 \\ g & 0 & 0 & \epsilon_1 \epsilon_2 \end{pmatrix}, \quad Z = R_{3.1}(\epsilon_2, \epsilon_2, 1)$$

$$24. W = R_{3.1}(\epsilon, \epsilon, 1), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 1 & 0 \\ 0 & 0 & 1 & 0 \\ g & 0 & 0 & \epsilon \end{pmatrix}, \quad Z = R_{0.1}$$

$$25. W = R_{3.1}(\epsilon, \epsilon, 1), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 1 & 0 \\ 0 & 0 & -1 & 0 \\ g & 0 & 0 & -\epsilon \end{pmatrix}, \quad Z = R_{0.2}$$

$$26. W = R_{3.1}(-1, -1, 1), \quad X = \begin{pmatrix} 0 & ab & c & 0 \\ a & 0 & 0 & -c \\ d & 0 & 0 & b \\ 0 & -d & 1 & 0 \end{pmatrix}, \quad Z = P$$

$$27. W = R_{3.1}(1, 1, 1), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & b & 0 \\ c & 0 & d & 0 \\ g & c & h & d \end{pmatrix}, \quad Z = P$$

$$28. W = R_{3.1}(1, 1, 1), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & g & 0 & h \end{pmatrix}, \quad Z = P$$

$$29. W = R_{2.1}(r, s), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 1 & 0 & 0 & sa \end{pmatrix}, \quad Z = P$$

$$30. W = R_{2.1}(r, s), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 1 & 0 & 0 & sa \end{pmatrix}, \quad Z = R_{2.1}(a, (rsa)^{-1})$$

$$31. W = R_{2.1}(-1/s, s), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -s & 0 & 0 \\ 0 & 0 & a & 0 \\ 1 & 0 & 0 & sa \end{pmatrix}, \quad Z = R_{2.2}(a, x)$$

$$32. W = R_{2.1}(\epsilon, -\epsilon), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q & 0 & 0 & -\epsilon \end{pmatrix}, \quad Z = R_{1.2}(x)$$

$$33. W = R_{2.1}(s, s), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s^{-1} & 1 & 0 \\ 0 & 0 & a & 0 \\ g & 0 & 0 & sa \end{pmatrix}, \quad Z = P$$

$$34. W = R_{2.1}(i\epsilon_1, i\epsilon_1), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\epsilon_1 & 1 & 0 \\ 0 & 0 & \epsilon_2 & 0 \\ g & 0 & 0 & i\epsilon_1\epsilon_2 \end{pmatrix}, \quad Z = R_{2.1}(\epsilon_2, -\epsilon_2)$$

$$35. W = R_{2.1}(i\epsilon, i\epsilon), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\epsilon & q & 0 \\ 0 & 0 & 1 & 0 \\ a & 0 & 0 & i\epsilon \end{pmatrix}, \quad Z = R_{1.2}(-1 + \frac{2}{aq})$$

$$36. W = R_{2.1}(1, -1), \quad X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 1 & 0 & 0 & b \\ 0 & -1 & b & 0 \end{pmatrix}, \quad Z = P$$

$$37. W = R_{2.1}(1, -1), \quad X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 1 & b & 0 \\ 1 & 0 & 0 & -b \end{pmatrix}, \quad Z = R_{2.2}(b/a, b/a)$$

$$38. W = R_{2.1}(1, -1), \quad X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 1 & b & 0 \\ 1 & 0 & 0 & -b \end{pmatrix}, \quad Z = R_{1.1}(a/b)$$

$$39. W = R_{2.1}(\epsilon, -\epsilon), \quad X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \epsilon a & 0 & 0 \\ 0 & \sqrt{\frac{a-b}{\epsilon a}} & b & 0 \\ \sqrt{\frac{\epsilon a}{a-b}} & 0 & 0 & -\epsilon b \end{pmatrix}, \quad Z = R_{1.2}(b/a)$$

$$40. W = R_{2.1}(1, -1), \quad X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 1/\sqrt{i} & b & 0 \\ \sqrt{i} & 0 & 0 & -b \end{pmatrix}, \quad Z = R_{0.3}$$

$$41. W = R_{2.2}(r, s), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 1 & 0 & 0 & -ar^{-1} \end{pmatrix}, \quad Z = P$$

$$42. W = R_{2.2}(r, s), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & 0 & a & 0 \\ 1 & 0 & 0 & -ar^{-1} \end{pmatrix}, \quad Z = R_{2.2}(a, x)$$

$$43. W = R_{2.2}(\epsilon, s), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q & 0 & 0 & -\epsilon \end{pmatrix}, \quad Z = R_{1.2}(x)$$

$$44. W = R_{2.2}(s, s), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s^{-1} & 1 & 0 \\ 0 & 0 & a & 0 \\ g & 0 & 0 & -a/s \end{pmatrix}, \quad Z = P$$

$$45. \ W = R_{2.2}(s, s), \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s^{-1} & q & 0 \\ 0 & 0 & 1 & 0 \\ g & 0 & 0 & -1/s \end{pmatrix}, \ Z = R_{1.2}(\frac{s^2-1}{qgs^2} - 1)$$

$$46. \ W = R_{1.1}(r), \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & 1 & 0 \\ 0 & 0 & -\epsilon & 0 \\ \epsilon & 0 & 0 & \epsilon r \end{pmatrix}, \ Z = P$$

$$47. \ W = R_{1.1}(r), \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & q & 0 \\ 0 & 0 & 1 & 0 \\ -q & 0 & 0 & -r \end{pmatrix}, \ Z = R_{1.2}(\frac{r^2-1}{q^2} - 1)$$

$$48. \ W = R_{1.1}(r), \ X = \begin{pmatrix} \epsilon a \frac{r+1}{r-1} & \epsilon c & 1 & 0 \\ \epsilon c & \epsilon a \frac{r+1}{r-1} & 0 & -1 \\ \epsilon & 0 & a & c \frac{r+1}{r-1} \\ 0 & -\epsilon & c \frac{r+1}{r-1} & a \end{pmatrix}, \ c^2 = a^2 - \frac{r-1}{r+1},$$

$$Z = P$$

$$49. \ W = R_{1.1}(r), \ X = \begin{pmatrix} \epsilon(a \frac{r+1}{r-1} + c) & 0 & 0 & 1 \\ 0 & \epsilon(a \frac{r+1}{r-1} - c) & 1 & 0 \\ 0 & \epsilon & (a + c \frac{r+1}{r-1}) & 0 \\ \epsilon & 0 & 0 & (a - c \frac{r+1}{r-1}) \end{pmatrix},$$

$$c^2 = a^2 - \frac{r-1}{r+1}, \ Z = R_{1.1}(\epsilon a/c),$$

$$50. \ W = R_{1.1}(i), \ X = \begin{pmatrix} a & 1-a^2 & 0 & 0 \\ 1 & -a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ Z = P$$

$$51. \ W = R_{1.1}(i), \ X = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 1 & -\epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ Z = R_{0.1}$$

$$52. \ W = R_{1.1}(i), \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ Z = R_{1.4}(x)$$

$$53. \ W = R_{1.1}(i), \ X = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ Z = R_{1.4}(x)$$

$$54. \quad W = R_{1.2}(s), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 1 & 0 & 0 & -\epsilon \end{pmatrix}, \quad Z = P$$

$$55. \quad W = R_{1.2}(s), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q & 0 & 0 & -1 \end{pmatrix}, \quad Z = R_{1.2}(x)$$

$$56. \quad W = R_{1.2}(s), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad Z = R_{1.2}^T(x)$$

$$57. \quad W = R_{1.2}(s), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s^{-1} & s-1 & 0 \\ 0 & 0 & \epsilon & 0 \\ \epsilon & 0 & 0 & -\epsilon s^{-1} \end{pmatrix}, \quad Z = P$$

$$58. \quad W = R_{1.2}(s), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s^{-1} & a(s-1) & 0 \\ 0 & 0 & 1 & 0 \\ a & 0 & 0 & -s^{-1} \end{pmatrix}, \quad Z = R_{1.2}(\frac{s+1}{a^2 s^2} - 1)$$

$$59. \quad W = R_{1.2}(s), \quad X = \begin{pmatrix} s^{-1} & 0 & 0 & a(1-s) \\ 0 & 1 & 0 & 0 \\ 0 & a & s^{-1} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad Z = R_{1.2}^T(\frac{s+1}{a^2 s^2} - 1)$$

$$60. \quad W = R_{1.2}(s), \quad X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & a^2 - s - 1 & 0 \end{pmatrix}, \quad Z = P$$

$$61. \quad W = R_{1.2}(s), \quad X = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 1/\sqrt{i} & \sqrt{a^2 - s - 1} & 0 \\ \sqrt{i} & 0 & 0 & -\sqrt{a^2 - s - 1} \end{pmatrix},$$

$$Z = R_{0.3}$$

$$62. \quad W = R_{1.3}(u), \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & b - u - 1 & -ua & 1 \end{pmatrix}, \quad Z = P$$

$$63. \ W = R_{1.3}(u), \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/b & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & b-u-1 & u/b & 1 \end{pmatrix}, \ Z = R_{1.3}(\frac{u+1}{b}-1)$$

$$64. \ W = R_{1.3}(1), \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 1 & 0 \\ a+b+1 & 0 & 1 & 0 \\ 0 & a+b-1 & b & 1 \end{pmatrix}, \ Z = P$$

$$65. \ W = R_{1.4}(t), \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & \epsilon & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \ Z = P$$

$$66. \ W = R_{1.4}(t), \ X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \\ 1 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \end{pmatrix}, \ Z = P$$

$$67. \ W = R_{1.4}(t), \ X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \ Z = R_{1.4}(x)$$

$$68. \ W = R_{1.4}(t), \ X = \begin{pmatrix} a & 1-a^2 & 0 & 0 \\ 1 & -a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ Z = P$$

$$69. \ W = R_{1.4}(t), \ X = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 1 & -\epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ Z = R_{0.1}$$

$$70. \ W = R_{1.4}(t), \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ Z = R_{1.4}(x)$$

$$71. \ W = R_{1.4}(t), \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \ Z = R_{1.4}(x)$$

$$72. \ W = R_{1.4}(t), \ X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ a & 1 & 0 & 0 \\ 1-a^2 & -a & 0 & 0 \end{pmatrix}, \ Z = P$$

$$73. \ W = R_{1.4}(t), \ X = \begin{pmatrix} 0 & 0 & 0 & p \\ 0 & 0 & 1/p & 0 \\ 0 & \epsilon/p & 0 & 0 \\ \epsilon p & 0 & 0 & 0 \end{pmatrix}, \ Z = R_{1.4}(x)$$

$$74. \ W = R_{1.4}(t), \ X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & x & -1 \\ 1 & 0 & 0 & 0 \\ -x & -1 & 0 & 0 \end{pmatrix}, \ Z = R_{0.2}$$

$$75. \ W = R_{1.4}(x), \ X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i\epsilon & 0 \\ 0 & i\epsilon & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ Z = R_{1.4}(x)$$

$$76. \ W = R_{1.4}(x), \ X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i\epsilon & 0 \\ 0 & i\epsilon & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ Z = R_{0.3}$$

$$77. \ W = R_{0.1}, \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ b & 0 & \epsilon & 0 \\ 0 & c & 0 & -a \end{pmatrix}, \ Z = P$$

$$78. \ W = R_{0.1}, \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ c & 0 & -1 & 0 \\ a & c & -b & -1 \end{pmatrix}, \ Z = P$$

$$79. \ W = R_{0.1}, \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & -1 & 0 \\ a & c & 0 & -1 \end{pmatrix}, \ Z = R_{0.2}$$

$$80. \ W = R_{0.2}, \ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 1 & 0 & 0 & -\epsilon \end{pmatrix}, \ Z = P$$

$$81. \quad W = R_{0.2}, \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ q & 0 & 0 & 1 \end{pmatrix}, \quad Z = R_{0.2}$$

$$82. \quad W = R_{0.2}, \quad X = \begin{pmatrix} 0 & \epsilon b & 0 & 0 \\ \epsilon & 0 & 0 & 0 \\ 1 & 0 & 0 & b \\ 0 & -1 & 1 & 0 \end{pmatrix}, \quad Z = P$$

$$83. \quad W = R_{0.3}, \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & -1 & 0 \\ i & 0 & 0 & a \end{pmatrix}, \quad Z = P$$

$$84. \quad W = R_{0.3}, \quad X = \begin{pmatrix} a & b & 1 & 0 \\ b & a & 0 & -1 \\ 0 & i & -b & -a \\ -i & 0 & -a & -b \end{pmatrix}, \quad Z = P$$

$$85. \quad W = R_{0.3}, \quad X = \begin{pmatrix} a & 0 & 0 & i \\ 0 & a & 1 & 0 \\ 0 & 1 & -a & 0 \\ i & 0 & 0 & a \end{pmatrix}, \quad Z = R_{0.3}$$

$$86. \quad W = R_{0.4} = P, \quad X = \text{arbitrary matrix}, \quad Z = P$$

4 Conclusion

It is possible to solve completely the system of equations (6)–(9) in the dimension two with the assistance of the computer programs for symbolic manipulations. Very important tool for both solving and classification of solutions is the use of symmetries.

The invertible solutions of the WXZ system are classified in this paper and the authors believe that that they were careful enough so that the solution is complete up to the symmetries of the system (12)–(28). The rigorous proof of the completeness of the solution set is rather difficult task. It is not known for pure Yang–Baxter equation either to the best knowledge of the authors.

The number of the solutions is rather large so that it is not possible to investigate in detail all the quantum doubles generated by the Vladimirov's procedure. It is therefore necessary to formulate supplementary conditions for the solutions that guarantee some suitable properties of the quantum doubles. The minimal condition that can be imposed for this goal is that both W and Z are second invertible as well [15].

The results of the subsection 3.2, can be reinterpreted as two-dimensional representations of the algebras

$$W_{12}T_1T_2 = T_2T_1W_{12} \quad (68)$$

for any invertible solution W of the Yang–Baxter equation.

5 Appendices

5.1 List of invertible solutions of the Yang–Baxter equation

The important starting point for solving the WXZ system is the knowledge of complete set of invertible solutions of the constant Yang–Baxter equation $[R, R, R] = 0$. Up to the symmetries of the solution set of the Yang–Baxter equation

$$R' = \rho(T \otimes T)R(T \otimes T)^{-1}, \quad R' = R^+, \quad R' = R^T, \quad (69)$$

it consists [6, 2, 3] of the following (intersecting) subsets parametrized by complex numbers r, s, t, u, v ,

Solution with three parameters

$$R_{3.1}(r, s, t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & t \end{pmatrix}, \quad rst \neq 0$$

Solutions with two parameters

$$\begin{aligned} R_{2.1}(r, s) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 1 - rs & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad rs \neq 0 \\ R_{2.2}(r, s) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 1 - rs & s & 0 \\ 0 & 0 & 0 & -rs \end{pmatrix}, \quad rs \neq 0 \end{aligned}$$

$$R_{2.3}(u, v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ u & 0 & 1 & 0 \\ v & u & 1 & 1 \end{pmatrix}$$

Solutions with one parameter

$$R_{1.1}(r) = \begin{pmatrix} r - r^{-1} + 2 & 0 & 0 & r - r^{-1} \\ 0 & r + r^{-1} & r - r^{-1} & 0 \\ 0 & r - r^{-1} & r + r^{-1} & 0 \\ r - r^{-1} & 0 & 0 & r - r^{-1} - 2 \end{pmatrix}, \quad r \neq 0$$

$$R_{1.2}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 - s & s & 0 \\ 1 & 0 & 0 & -s \end{pmatrix}, \quad s \neq 0$$

$$R_{1.3}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ u & -u & u & 1 \end{pmatrix}$$

$$R_{1.4}(t) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & t & 0 \\ 0 & t & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad t \neq 0$$

Solutions without parameter

$$R_{0.1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$R_{0.2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$R_{0,3} = \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}$$

$$R_{0,4} = P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5.2 Equivalence classes of the 4×4 matrices with respect to the transformation $X' = (\mathbf{1} \otimes S)X(\mathbf{1} \otimes S)^{-1}$

Lemma: Any 4×4 matrix A can be transformed into one of following 14 forms using transformation

$$A' = \lambda(\mathbf{1} \otimes S)A(\mathbf{1} \otimes S)^{-1} \quad (70)$$

where $\lambda \in \mathbf{C}$, $\mathbf{1} = \text{diag}(1, 1)$, $S \in \mathcal{GL}(2, \mathbf{C})$.

$$A_1 = \begin{pmatrix} a1 & a2 & \alpha + 1 & 0 \\ b1 & b2 & 0 & \alpha - 1 \\ c1 & c2 & c3 & c4 \\ d1 & d2 & d3 & d4 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} a1 & a2 & a3 & 0 \\ b1 & b2 & 1 & a3 \\ c1 & 1 & c3 & c4 \\ 0 & d2 & d3 & d4 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} a1 & a2 & a3 & 0 \\ b1 & b2 & 1 & a3 \\ \alpha + 1 & 0 & c3 & c4 \\ 0 & \alpha - 1 & d3 & d4 \end{pmatrix}$$

$$\begin{aligned}
A_4 &= \begin{pmatrix} a1 & a2 & a3 & 0 \\ b1 & b2 & 1 & a3 \\ c1 & 0 & c3 & c4 \\ d1 & c1 & d3 & d4 \end{pmatrix} \\
A_5 &= \begin{pmatrix} a1 & a2 & a3 & 0 \\ b1 & b2 & 0 & a3 \\ \alpha + 1 & 0 & c3 & c4 \\ 0 & \alpha - 1 & d3 & d4 \end{pmatrix} \\
A_6 &= \begin{pmatrix} a1 & a2 & a3 & 0 \\ b1 & b2 & 0 & a3 \\ c1 & 0 & c3 & 1 \\ 1 & c1 & 0 & d4 \end{pmatrix} \\
A_7 &= \begin{pmatrix} a1 & a2 & a3 & 0 \\ b1 & b2 & 0 & a3 \\ c1 & 0 & \alpha + 1 & 0 \\ 1 & c1 & 0 & \alpha - 1 \end{pmatrix} \\
A_8 &= \begin{pmatrix} a1 & a2 & a3 & 0 \\ b1 & b2 & 0 & a3 \\ c1 & 0 & c3 & 0 \\ 1 & c1 & d3 & c3 \end{pmatrix} \\
A_9 &= \begin{pmatrix} a1 & a2 & a3 & 0 \\ b1 & b2 & 0 & a3 \\ c1 & 0 & \alpha + 1 & 0 \\ 0 & c1 & 0 & \alpha - 1 \end{pmatrix} \\
A_{10} &= \begin{pmatrix} a1 & 1 & a3 & 0 \\ 0 & b2 & 0 & a3 \\ c1 & 0 & c3 & 0 \\ 0 & c1 & 1 & c3 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
A_{11} &= \begin{pmatrix} \alpha + 1 & 0 & a3 & 0 \\ 0 & \alpha - 1 & 0 & a3 \\ c1 & 0 & c3 & 0 \\ 0 & c1 & 1 & c3 \end{pmatrix} \\
A_{12} &= \begin{pmatrix} a1 & 0 & a3 & 0 \\ b1 & a1 & 0 & a3 \\ c1 & 0 & c3 & 0 \\ 0 & c1 & 1 & c3 \end{pmatrix} \\
A_{13} &= \begin{pmatrix} a1 & 0 & a3 & 0 \\ 0 & b2 & 0 & a3 \\ c1 & 0 & c3 & 0 \\ 0 & c1 & 0 & c3 \end{pmatrix} \\
A_{14} &= \begin{pmatrix} a1 & 0 & a3 & 0 \\ 1 & a1 & 0 & a3 \\ c1 & 0 & c3 & 0 \\ 0 & c1 & 0 & c3 \end{pmatrix}
\end{aligned}$$

Proof: In order to prove previous lemma it is convenient to write matrix A divided into square blocks, i.e.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \quad (71)$$

Blocks A_{11}, \dots, A_{22} are transforming like $A'_{ij} = \lambda S A_{ij} S^{-1}$ under transformation (70). It is possible to convert one chosen block into Jordan's canonical form, i.e. one of the following forms:

$$1) \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \quad 2) \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix} \quad 3) \begin{pmatrix} \alpha + 1 & 0 \\ 0 & \alpha - 1 \end{pmatrix} \quad (72)$$

(see Jordan's theorem in linear algebra). Case 1) is of course invariant w.r.t. (70), in case 2) it is possible to convert another arbitrary block into upper triangular (with non-diagonal element equal 1) or lower triangular (with identical diagonal elements) or diagonal block. Using these properties, one can find given list of classes, firstly simplifying upper right block, secondly using remaining symmetries to simplify lower left block, then lower right block and finally upper left block.

5.3 List of X matrices

Solutions of the equation (8) can be written in terms of the following matrices.

$$\begin{aligned}
X_1(a, b, c) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \\
X_2(a, b, c) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & c & a \end{pmatrix}, \quad b \neq 0 \text{ or } c \neq 0 \\
X_3(a, b, c) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & c & 0 & a \end{pmatrix}, \\
X_4(a, b, c) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & b & a & 1 \end{pmatrix}, \quad a \neq 0 \text{ or } b \neq 0 \text{ or } c \neq 0 \\
X_5(a, b, c) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \end{pmatrix}, \quad abc \neq 0 \\
X_6(a, b, c) &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \\ b & 0 & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix}, \quad abc \neq 0 \\
X_7(a, b) &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & a & 1 \\ 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \end{pmatrix}, \quad a \neq 0 \text{ or } b \neq 0 \\
X_8(a) &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & a & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
X_9(b, c, d) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 1 & 0 & 0 & d \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
X_{10}(b, c, d, g) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & c & 0 \\ g & 0 & 0 & d \end{pmatrix} \quad g \neq 0 \\
X_{11}(a, b, c, d) &= \begin{pmatrix} 0 & ab & 1 & 0 \\ a & 0 & 0 & -1 \\ d & 0 & 0 & b \\ 0 & -d & 1 & 0 \end{pmatrix}, \\
X_{12}(a, b) &= \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 1 & 0 & 0 & b \\ 0 & -1 & b & 0 \end{pmatrix} \\
X_{13}(g) &= \begin{pmatrix} 0 & 0 & i+1 & 0 \\ 2(i+1)g & 0 & 0 & i-1 \\ -ig & 0 & 0 & 1 \\ 0 & g & 0 & 0 \end{pmatrix} \\
X_{14}(a, \epsilon, p) &= \begin{pmatrix} \epsilon ap & \epsilon c & 1 & 0 \\ \epsilon c & \epsilon ap & 0 & -1 \\ \epsilon & 0 & a & cp \\ 0 & -\epsilon & cp & a \end{pmatrix}, \quad c^2 p = a^2 p - 1 \\
X_{15}(a) &= \begin{pmatrix} a & 1-a^2 & 0 & 0 \\ 1 & -a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad a \neq \pm 1 \\
X_{16}(a, b, c, d) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & c & d & 1 \end{pmatrix}, \quad a \neq d, b \neq c \\
X_{17}(a, b) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 1 & 0 \\ a+b+1 & 0 & 1 & 0 \\ 0 & a+b-1 & b & 1 \end{pmatrix} \\
X_{18}(a) &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ a & 1 & 0 & 0 \\ 1-a^2 & -a & 0 & 0 \end{pmatrix} \\
X_{19}(a, b, c, \epsilon) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ b & 0 & \epsilon & 0 \\ 0 & c & 0 & -a \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
X_{20}(a, b, c) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ c & 0 & -1 & 0 \\ a & c & -b & -1 \end{pmatrix} \\
X_{21}(a, b) &= \begin{pmatrix} a & b & 1 & 0 \\ b & a & 0 & -1 \\ 0 & i & -b & -a \\ -i & 0 & -a & -b \end{pmatrix} \\
X_{22}(a, b, c, d, e, f, g, h) &= \begin{pmatrix} a & 0 & b & 0 \\ c & a & d & b \\ e & 0 & f & 0 \\ g & e & h & f \end{pmatrix} \\
X_{23}(a, b, c, d, e, f, g, h) &= \begin{pmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ e & 0 & f & 0 \\ 0 & g & 0 & h \end{pmatrix}
\end{aligned}$$

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